Overcoming Element Quality Dependency with Adaptive Extended Stencil FEM (AES-FEM)

Rebecca Conley, Tristan J. Delaney, Xiangmin Jiao
Stony Brook University

Introduction: Problem Formulation

Finite element methods (FEM) are powerful tools for solving partial differential equations in many scientific and engineering fields. FEMs are highly dependent on mesh quality. We propose a method called Adaptive Extended Stencil FEM (AES-FEM), which replaces the traditional basis functions with polynomials constructed via a local weighted least squares approximations over an adaptively selected stencil. The test functions are the standard hat functions. AES-FEM performs better on meshes with poor element quality than traditional FEM.

Methodology: AES-FEM Description

We start with the weak form of a PDE; for simplicity, consider a Poisson equation. Using GLP basis functions and hat test functions, the discretized weak form becomes:

\[-\sum_{j=1}^{n} u_j \int_{\Omega} \nabla \psi_j \cdot \nabla \psi_i \, dV = \int_{\Omega} f \psi_i \, dV\]

The order of convergence is controlled by the degree of the GLP basis functions.

To construct the GLP basis functions, we start with the Taylor series expansion and form a linear system. Utilizing a row scaling matrix, the points closer to the center of the stencil are assigned a heavier weight. A column scaling matrix improves stability if the system is ill-conditioned. The resulting system is solved within a least-squares formulation, using truncated QR with column pivoting. The GLP basis functions form a partition of unity. AES-FEM has the advantage that high degree GLP basis functions can be constructed from traditional “linear” meshes to guarantee high-order convergence, avoiding the challenges of meshing with isoparametric high-order elements for complex geometries.

Numerical Results

We have introduced a generalization of the finite element method called AES-FEM which is less dependent on element quality than traditional FEM. The generalized Lagrange polynomial (GLP) basis functions are used instead of the traditional basis functions, and the standard hat functions are used as test functions. The condition number of the stiffness matrix of AES-FEM does not increase when used on a mesh with poor quality elements. The accuracy of AES-FEM is better than FEM in the experiments we performed. Future work includes improving efficiency, exploring the use of multilevel solver methods, and applications to moving-boundary problems.

Conclusions

Goal: Decrease Dependency on Element Quality

As the mesh quality degrades, the condition number of the FEM stiffness matrix can become arbitrarily large, leading to slow convergence of iterative solvers and even inaccurate solutions. We overcome these issues by introducing generalized Lagrange polynomial (GLP) basis functions on an extended neighborhood, and then propose AES-FEM to improve the accuracy and stability of FEM.

Figure 1: As the element quality degrades, the condition number increases for FEM.

Figure 2: As the element quality degrades, the iterations number remains the same for AES-FEM.

Figure 3: Examples of two poorly shaped triangles and a sliver tetrahedron.

Figure 4: Example of unstructured mesh of unit cube.

Figure 5: Errors for 3D Poisson equation on the unit cube.

Figure 6: Errors for 3D convection-diffusion equation on the unit cube.

Figure 7: Errors for 2D convection-diffusion equation on unit circle.

Acknowledgements

Partially supported by Army Research Office under Grant W911NF1310249.

References
